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APPROXIMATE INTERVAL ESTIMATES FOR
MECHANICAL RELIABILITY

by
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Thesis Advisor: W. Max Woods

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Two approximate interval estimation procedures for mechanical component reliability, $P(X > Y)$, are developed and their accuracy evaluated by computer simulations. The strength, X, of the component and the stress, Y, applied to it are independent normally distributed variables with unknown means and variances. In the first interval procedure the variances are equal. In the second procedure the variances may be unequal.

The derived intervals are quite accurate for the cases simulated which include large and small sample sizes. These procedures are simple to apply and require the use of percentile points of the Student's t distribution. In the second procedure, the degrees of freedom of the associated t statistic is a function of the test data , and therefore it is random.

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Approximate Interval Estimates for Mechanical Reliability

by

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ABSTRACT

Two approximate interval estimation procedures for mechanical component reliability, $P(X > Y)$, are developed and their accuracy evaluated by computer simulations. The strength, X, of the component and the stress, Y, applied to it are independent normally distributed variables with unknown means and variances. In the first interval procedure the variances are equal. In the second procedure the variances may be unequal.

The derived intervals are quite accurate for the cases simulated which include large and small sample sizes. These procedures are simple to apply and require the use of percentile points of the Student's t distribution. In the second procedure, the degrees of freedom of the associated t statistic is a function of the test data , and therefore it is random.

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I. INTRODUCTION

Let X and Y be independent random variables with normal cumulative distribution functions $F_X(x)$ and $F_Y(y)$ respectively. Suppose X is the strength of a mechanical component, and Y is the stress applied to the component. The strength depends on the material properties, manufacturing procedures and other factors. The stress is a function of the environment to which the component is subjected. Component failure is defined by the event $X \leq Y$. Component reliability, R, is defined by $R = P(X > Y)$. R is called mechanical reliability. Two lower confidence limit procedures for R are developed in this thesis. In both procedures the means and variances of X and Y are unknown. In one procedure the variances are assumed to be equal.

A nonparametric interval estimation procedure for R was first proposed by Birnbaum [Ref. 1: pp. 13-17] using the Mann-Whitney U statistic. Birnbaum and McCarty developed a procedure for computing the minimum sample size needed for such a confidence interval to have a given width and confidence level [Ref. 2: pp. 558-562]. Owen, Craswell and Hanson [Ref. 3: pp. 906-924] provided more detailed tables for use in computing sample sizes and confidence intervals for the Birnbaum-McCarty procedure. Tables designed especially for the normal distribution were also included in their paper. Govindarajulu [Ref. 4: pp. 229-238] observed that the bounds employed by Birnbaum and McCarty for obtaining the confidence intervals can be substantially improved asymptotically and reduced the Birbaum-McCarty bounds by approximately 1.2. Church and Harris [Ref. 5: pp. 49-54] pointed out that the sample sizes required by these nonparametric procedure are likely to be too large for many practical situations.

Owen [Ref. 6: pp.445-478] gave an exact confidence limit procedure for $R = P(X \geq x)$ where X is normally distributed with unknown mean and unknown variance, and x is a constant. His procedure uses the noncentral t distribution and extensive table lookups are needed. Owen and Hua [Ref. 7: pp. 285-311] developed special tables that reduced these calculations. Their tables were limited to two confidence level values; namely 90% and 95%. Lee [Ref. 8: pp. 15-22] reports on a closed form equation for this confidence limit that is approximate but quite accurate. His equation applies for any confidence level and uses the central student's t distribution.

Church and Harris [Ref. 5: pp. 49-54] developed approximate confidence limits for $R = P(X > Y)$, under the assumption that the stress, Y, has a standard normal distribution and $F_X(x)$ is normally distributed with unknown mean and variance.

Throughout this thesis we write $X \sim N(\mu, \sigma^2)$ to denote that X has a normal distribution with mean μ and variance σ^2.

II. APPROXIMATE INTERVAL ESTIMATION PROCEDURE FOR RELIABILITY R = P(X > Y) — EQUAL VARIANCE CASE

A. LOWER CONFIDENCE LIMIT PROCEDURE

Suppose the strength, X, of a mechanical device and the stress, Y, applied to it are independent variables, with normal probability distributions. We assume both means are unknown and both variances are unknown but have common values σ^2. The mechanical reliability, R, of the device is defined as follows:

$$\begin{aligned} R &= P[X > Y] \\ &= P\left[\frac{X - Y - (\mu_X - \mu_Y)}{\sigma\sqrt{2}} > -\frac{\mu_X - \mu_Y}{\sigma\sqrt{2}}\right] \\ &= \Phi\left(\frac{\mu_X - \mu_Y}{\sigma\sqrt{2}}\right). \end{aligned} \tag{2.1}$$

where Φ is the standard normal cumulative distribution function. Let $\delta = \frac{\mu_X - \mu_Y}{\sigma}$. Then

$$R = \Phi\left(\frac{\delta}{\sqrt{2}}\right). \tag{2.2}$$

A consistent estimator [Ref. 9: PP.289-294] of δ is

$$K = \frac{\overline{X} - \overline{Y}}{S}; \tag{2.3}$$

where

$$S = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2 + \sum_{j=1}^{m} (Y_j - \overline{Y})^2}{n + m - 2}}$$

is the pooled sample variance of X and Y; n is the size of the sample on X; m is the size of the sample on Y; and $\overline{X}$ and $\overline{Y}$ are the respective sample means. Since X and Y are independently normally distributed,

$$\overline{X} - \overline{Y} \sim N(\mu_X - \mu_Y, \sigma^2(\frac{1}{n} + \frac{1}{m}))$$

The general method for deriving confidence intervals [Ref. 9: PP.347-355] can be used to obtain a lower $100(1-\alpha)\%$ confidence limit, $\hat{\delta}_{L,1-\alpha}$, for δ. Suppose $\frac{\overline{x}-\overline{y}}{s}$ is constructed from the data. Then under the general method, $\hat{\delta}_{L,1-\alpha}$ is the value of δ such that

$$\begin{aligned}
1-\alpha &= P\left[\frac{\overline{X}-\overline{Y}}{S} \le \frac{\overline{x}-\overline{y}}{s}\right] \\
&= P\left[\frac{\dfrac{\overline{X}-\overline{Y}-(\mu_X-\mu_Y)+(\mu_X-\mu_Y)}{\sigma\sqrt{1/n+1/m}}}{\dfrac{S}{\sigma}} \le \frac{\overline{x}-\overline{y}}{s\sqrt{1/n+1/m}}\right] \\
&= P\left[\frac{Z+\dfrac{\mu_X-\mu_Y}{\sigma\sqrt{1/n+1/m}}}{\sqrt{\dfrac{\chi^2_{n-m-2}}{n+m-2}}} \le \frac{\overline{x}-\overline{y}}{s\sqrt{1/n+1/m}}\right] \\
&= P\left[N.C.T.\left(\frac{\mu_X-\mu_Y}{\sigma\sqrt{1/n+1/m}}\right) \le \frac{\overline{x}-\overline{y}}{s\sqrt{1/n+1/m}}\right] \\
&= P\left[N.C.T.\left(\frac{\delta}{\sqrt{1/n+1/m}}\right) \le \frac{\kappa}{\sqrt{1/n+1/m}}\right],
\end{aligned} \tag{2.4}$$

where N.C.T. denotes a noncentral t random variable with noncentrality parameter

$$\frac{\delta}{\sqrt{1/n+1/m}}, \ \delta = \frac{\mu_X-\mu_Y}{\sigma} \text{ and } \kappa = \frac{\overline{x}-\overline{y}}{s}.$$

The degrees of freedom of this noncentral t is n+m-2. One can use the noncentral t table to obtain the solution for δ in Equation (2.4) which is $\hat{\delta}_{L,1-\alpha}$. Owen and Hua [Ref.

7: pp. 285-311] have taken this approach for the univariate case. Their method required the extensive use of tables. We shall take a different approach here, in order to find a closed expression for an approximation to $\hat{\delta}_L$.

In his thesis, Lee [Ref. 8: pp. 15-22] developed an approximate $100(1-\alpha)\%$ lower confidence limit for $R = P(X > x)$ where $X \sim N(\mu, \sigma^2)$, μ and σ^2 are unknown, and x a constant. Lee's expression is

$$\hat{\delta}_{L,\,1-\alpha} = K_x - \left[\frac{1}{n} + \frac{K_x^2}{2(n - \sqrt{K_x})} \right]^{1/2} t_{1-\alpha,\, n-1}, \tag{2.5}$$

where $K_x = \dfrac{\overline{X} - x}{S}$, and $t_{1-\alpha, n-1}$ is the $100(1-\alpha)^{\text{th}}$ percentile point of the student's t distribution with n-1 degrees of freedom.

A method analogous to Lee's procedure can be used to develop an equation for $\hat{\delta}_{L,1-\alpha}$ in the bivariate case described at the outset of this section. If we substitute $(1/n + 1/m)$ for $(1/n)$ and $(n + m - 1)$ for (n) in Equation (2.5), we would obtain the lower confidence limit

$$\hat{\delta}_{L,\,1-\alpha} = K - \left[\left(\frac{1}{n} + \frac{1}{m} \right) + \frac{K^2}{2(n + m - 1 - \sqrt{k})} \right]^{1/2} t_{1-\alpha,\, n-m-2}. \tag{2.6}$$

where $K = \dfrac{\overline{X} - \overline{Y}}{S}$. The corresponding $100(1-\alpha)\%$ lower confidence limit for $R = P(X > Y)$ is :

$$\hat{R}_{L,\,1-\alpha} = \Phi\left(\frac{\hat{\delta}_{L,1-\alpha}}{\sqrt{2}} \right). \tag{2.7}$$

Our computer simulation results show that this lower confidence limit is quite accurate. The results are tabulated in Tables 1 and 2. A description of the simulation procedure along with a analysis on computer results will be given later in this chapter. The important point of these tables is the comparison between R and $\hat{R}_{100(1-\alpha),\,L(1-\alpha)}$. If these two

values are equal, the approximate lower confidence limit procedures given in Equations (2.6) and (2.7) are nearly exact.

Table 1. ANALYSIS OF 90% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR EQUAL VARIANCES CASE WITH EQN. 2.6

R	σ	n	m	$\hat{R}_{1000(1-\alpha),L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	8	8	.8928	.9130	.1398	.0110
		8	30	.9047	.8880	.0990	.0068
		20	30	.9033	.8890	.0704	.0033
	20.0	8	8	.8928	.9130	.1398	.0110
		8	30	.9047	.8880	.0990	.0068
		20	30	.9033	.8890	.0704	.0033
.950	1.0	8	8	.9450	.9130	.1092	.0073
		8	30	.9527	.8870	.0722	.0040
		20	30	.9515	.8900	.0508	.0019
	20.0	8	8	.9451	.9130	.1092	.0073
		8	30	.9527	.8870	.0721	.0040
		20	30	.9516	.8900	.0508	.0019
.990	1.0	8	8	.9877	.9200	.0563	.0025
		8	30	.9908	.8820	.0310	.0009
		20	30	.9903	.8960	.0212	.0004
	20.0	8	8	.9877	.9200	.0563	.0025
		8	30	.9908	.8820	.0310	.0009
		20	30	.9903	.8960	.0212	.0004

Table 2. ANALYSIS OF 95% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR EQUAL VARIANCES CASE WITH EQN. 2.6

R	σ	n	m	$\hat{R}_{1000(1-z), L(1-z)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	8	8	.8855	.9650	.1954	.0126
		8	30	.8983	.9530	.1332	.0078
		20	30	.9013	.9470	.0938	.0038
	20.0	8	8	.8855	.9650	.1954	.0126
		8	30	.8983	.9530	.1332	.0078
		20	30	.9013	.9470	.0938	.0038
.950	1.0	8	8	.9375	.9650	.1579	.0092
		8	30	.9486	.9510	.0990	.0049
		20	30	.9498	.9500	.0688	.0022
	20.0	8	8	.9376	.9650	.1578	.0092
		8	30	.9486	.9510	.0989	.0049
		20	30	.9498	.9500	.0688	.0022
.990	1.0	8	8	.9846	.9660	.0884	.0040
		8	30	.9898	.9510	.0447	.0014
		20	30	.9905	.9450	.0299	.0006
	20.0	8	8	.9847	.9660	.0883	.0040
		8	30	.9898	.9510	.0447	.0014
		20	30	.9905	.9450	.0298	.0006

The above method for deriving $\hat{\delta}_{L,1-\alpha}$ cannot be used to find a lower confidence interval for $P(X > Y)$ when we drop the assumption of equal variances. Consequently, we shall develop $\hat{\delta}_{L,1-\alpha}$ using a different approach which has greater potential for constructing confidence intervals when variances are not equal. Let

$$K = g(\overline{X} - \overline{Y},\ S^2) = \frac{\overline{X} - \overline{Y}}{\sqrt{S^2}} = \frac{\overline{X} - \overline{Y}}{S}. \tag{2.8}$$

The Taylor expansion of K at $(\mu_X - \mu_Y, \sigma^2)$, using only first order derivatives, is given by:

$$\begin{aligned} g(\overline{X} - \overline{Y},\ S^2) &= g(\mu_X - \mu_Y, \sigma^2) + [\overline{X} - \overline{Y} - (\mu_X - \mu_Y)] \frac{\partial g}{\partial(\overline{X} - \overline{Y})}\bigg|_{(\mu_X - \mu_Y, \sigma^2)} \\ &\quad + (S^2 - \sigma^2)\frac{\partial g}{\partial S^2}\bigg|_{(\mu_X - \mu_Y, \sigma^2)} + R_t \\ &= \frac{\mu_X - \mu_Y}{\sigma} + \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sigma} - (S^2 - \sigma^2)\frac{\mu_X - \mu_Y}{2\sigma^3} + R_t \end{aligned} \tag{2.9}$$

The expected value and variance of $g(\overline{X} - \overline{Y},\ S^2)$ are as follows:

$$E[K] = \mu_g \doteq \frac{\mu_X - \mu_Y}{\sigma}$$

$$\begin{aligned} \mathrm{Var}[K] &= \sigma_g^2 \\ &\doteq \frac{1}{\sigma^2}\mathrm{Var}(\overline{X} - \overline{Y}) + \left(\frac{\mu_X - \mu_Y}{2\sigma^3}\right)^2 \mathrm{Var}(S^2) \\ &= \frac{1}{\sigma^2}\left(\frac{\sigma^2}{n} + \frac{\sigma^2}{m}\right) + \frac{(\mu_X - \mu_Y)^2}{4\sigma^6}\,\frac{2\sigma^4}{n + m - 2} \\ &= \left(\frac{1}{n} + \frac{1}{m}\right) + \frac{(\mu_X - \mu_Y)^2}{2\sigma^2(n + m - 2)}; \end{aligned} \tag{2.10}$$

An estimator for σ_K, is then

$$\begin{aligned} \hat{\sigma}_K &= \sqrt{\left(\frac{1}{n} + \frac{1}{m}\right) + \frac{(\overline{X} - \overline{Y})^2}{2S^2(n + m - 2)}} \\ &= \sqrt{\left(\frac{1}{n} + \frac{1}{m}\right) + \frac{K^2}{2(n + m - 2)}}\ ; \end{aligned} \tag{2.11}$$

where $K = \frac{\bar{X} - \bar{Y}}{S}$.

For large sample size, n and m, the probability distribution of $\frac{K - \mu_K}{\hat{\sigma}_K}$ is close to the standard normal distribution. An approximate $100(1-\alpha)\%$ lower confidence limit for μ_K is $K - \hat{\sigma}_K Z_{1-\alpha}$, where $Z_{1-\alpha}$ is the $100(1-\alpha)\%$ percentile of the standard normal distribution. We choose to approximate the distribution of $\frac{K - \mu_K}{\hat{\sigma}_K}$ with a t distribution that has n+m-2 degrees of freedom. This approximation should accommodate small samples better than the normal approximation. The computer simulations will reveal the accuracy of this choice. Consequently an approximate lower confidence limit for μ_K is given by

$$\begin{aligned} \hat{\mu}_{K_{L,1-\alpha}} &= K - \hat{\sigma}_K t_{1-\alpha,\, n+m-2} \\ &= K - \left[\left(\frac{1}{n} + \frac{1}{m}\right) + \frac{K^2}{2(n+m-2)}\right]^{1/2} t_{1-\alpha,\, n+m-2} \\ &= \hat{\delta}_{L,\, 1-\alpha} \end{aligned} \tag{2.12}$$

The corresponding $100(1-\alpha)$ % lower confidence limit, $\hat{R}_{L,1-\alpha}$, of component reliability is

$$\hat{R}_{L,\,1-\alpha} = \Phi\left(\frac{\hat{\delta}_{L,\,1-\alpha}}{\sqrt{2}}\right). \tag{2.13}$$

1. Example

We illustrate the application of this procedure with an example. We compute the 90% lower confidence limit of the component reliability, $\hat{R}_{L,0.9}$ given the following data:

X: 9.26 10.19 9.79 11.27 10.06 9.22 9.75 8.46 9.79 9.52

11.55 9.41 9.99 9.22 11.22 9.89 9.54 9.24 8.77 9.86

10.03 10.33 10.95 8.33 10.85

Y: 8.07 8.53 7.74 [illegible] [illegible] 6.36 6.82 8.65 7.00 7.65

7.94 9.15 6.99 7.1[illegible] [illegible].88 9.56 9.58 7.85 8.91 7.52

9.97 8.76 7.01 9.96 [illegible] 9.05 9.54 10.72 7.86 7.87

$$n = 25, \quad \bar{x} = 9.860$$

$$m = 30, \quad \bar{y} = 8.308$$

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2}{n + m - 2}} = 0.978$$

$$\kappa = \frac{\bar{x} - \bar{y}}{s} = 1.587$$

$$\hat{\delta}_{L,\,0.9} = \kappa - \left[\left(\frac{1}{n} + \frac{1}{m}\right) + \frac{\kappa^2}{2(n + m - 2)}\right]^{1/2} t_{0.9,\; n+m-2}$$
$$= 1.\text{[illegible]}$$

$$\hat{R}_{L,\,0.9} = \Phi\left(\frac{\hat{\delta}_{L,\,0.9}}{\sqrt{2}}\right) = 0.798$$

We note that the point estimator of R,

$$\hat{R}\left(\frac{\mu_X - \mu_Y}{\sigma\sqrt{2}}\right) = \Phi\left(\frac{\kappa}{\sqrt{2}}\right) = \Phi(1.122) = 0.868,$$

and the computed 90% lower confidence limit of R, $\hat{R}_{L,0.9} = 0.798 < 0.868$ as it should be. We cannot draw any conclusion about the accuracy of any confidence interval procedure from one example. We need computer simulations to do this. The next section considers the accuracy of this procedure.

B. ACCURACY OF THE PROCEDURE

1. Measures of accuracy and the concept of computer simulation

The accuracy of the interval procedure in Equation (2.12) was evaluated in terms of the following four characteristics:

- The actual confidence level of the interval, i.e. the portion of times an estimated limit will cover the true reliability.
- The mean error between the estimated limit and the true reliability, which is denoted as 'mean error from R in the simulation result tables.
- The variance of the error between the estimated limit and true reliability, which is denoted as 'variance of error' in the tables.
- The $100(1-\alpha)^{th}$ percentile point of the distribution of $R_{L,1-\alpha}$.

The actual confidence level of an approximate confidence interval can be accessed during a computer simulation which we will discuss later.

To compute the $100(1-\alpha)^{th}$ percentile point of the $\hat{R}_{L,1-\alpha}$, one only need to examine the definition of $\hat{R}_{L,1-\alpha}$ as a lower $100(1-\alpha)$ % confidence limit for R, i.e.

$$P(\hat{R}_{L,1-\alpha} < R) \quad 1-\alpha. \tag{2.14}$$

This equation says that, R is the $100(1-\alpha)^{th}$ percentile point of the probability distribution of $\hat{R}_{L,1-\alpha}$. Thus, if we construct the distribution of $\hat{R}_{L,1-\alpha}$ by computer simulation, we should find that the $100(1-\alpha)^{th}$ percentile point of our constructed distribution is R, provided $\hat{R}_{L,1-\alpha}$ is a true $100(1-\alpha)$% lower confidence limit for R. Let $R^{*}_{L,1-\alpha}$ denote the $100(1-\alpha)^{th}$ percentile point of $\hat{R}_{L,1-\alpha}$. Then the quantity $|R^{*}_{L,1-\alpha} - R|$ is a measure of the accuracy of the procedure.

We can construct the distribution of $\hat{R}_{L,1-\alpha}$ by generating a large number, say 1000, of random observations on $\hat{R}_{L,1-\alpha}$ for a given set of parameter values $n, m, \mu_X, \mu_Y, \sigma^2$, and R. The $100(1-\alpha)$ empirical percentile point of the distribution of $R_{L,1-\alpha}$ is the $1000(1-\alpha)^{th}$ ordered statistic of $\hat{R}_{L,1-\alpha}$, $\hat{R}_{1000(1-\alpha),L(1-\alpha)}$.

2. Computer simulation

a. *Simulation procedure*

Eighteen sets of values of n, m, μ_X, μ_Y, and σ^2 were chosen to perform the simulations. They were selected in a manner so that $R = \Phi\left(\frac{\mu_X - \mu_Y}{2\sqrt{\sigma}}\right)$, for R = 0.90, 0.95, 0.99. Thus when random samples of X and Y are generated, the reliability $R = P(X > Y)$ will be at designed value. Parameters were also chosen according to the following rules in order to cover practical conditions:

- $\mu_X > \mu_Y$
- Reliability, R : 0.90, 0.95, 0.99
- Standard Deviations, σ: 1.0, 20.0
- Sample sizes, (n, m): (8, 8); (8, 30); (20, 30)

Actual sets of parameters are tabulated on Appendix B.

Each set of parameters describes a case. For each case, the set of parameters were used to generate samples of size n and m for normal variates X and Y respectively. The developed methods were used to estimate the confidence limit for confidence levels of $\alpha = 0.05$, 0.1, and 0.20. For each individual case, this procedure is replicated 1000 times producing 1000 random observations of $\hat{R}_{L,1-\alpha}$. The $1000(1-\alpha)^{th}$ ordered statistic of $\hat{R}_{L,1-\alpha}$ is compared with the true reliability. The actual confidence levels is computed by counting the number of the 1000 $\hat{R}_{L,1-\alpha}$ statistics that fall below the true reliability. We also computed the sample mean and variance of the estimation error using the 1000 generated values of $\hat{R}_{L,1-\alpha}$.

The simulation procedure can be summarized as follows:

1) Generate n random normal variates X, $X \sim N(\mu_X, \sigma^2)$; m random normal variates Y, $Y \sim N(\mu_Y, \sigma^2)$.
2) Compute $\overline{X}, \overline{Y}, S$.
3) Compute $K, \hat{\delta}_{L,1-\alpha}$.
4) Compute $\hat{R}_{L,1-\alpha} = \Phi\left(\frac{\hat{\delta}_{L,1-\alpha}}{\sqrt{2}}\right)$, for α = 0.05, 0.10, 0.20.

5) Repeat steps 1 through 4, for 1000 times. Then order the $\hat{R}_{L,1-\alpha}$ to get $\hat{R}_{(1),L(1-\alpha)}$, $\hat{R}_{(2),L(1-\alpha)}$, ..., $\hat{R}_{1000(1-\alpha),L(1-\alpha)}$, ..., $\hat{R}_{(1000),L(1-\alpha)}$, from the smallest to the largest.

6) Print $\hat{R}_{1000(1-\alpha),L(1-\alpha)}$.

7) Print $\max_i\{\hat{R}_{(i),L(1-\alpha)}: \hat{R}_{(i),L(1-\alpha)} \leq R\}$,the actual confidence level of the approximation will be $\frac{i}{1000}$.

8) Compute and print the sample mean and variance of the estimation error, to measure the precision and stability of the approximation.

b. Simulation language

The simulations were conducted on the N.P.S. mainframe IBM 3033 computer. The programming language used for the simulations is VS FORTRAN 2. The random number generator LLRANII was used to generate normal variates. The IMSL statistics function TIN was used to compute the percentile of the student t distribution and ANORDF was used to compute the probability of standard normal distribution. The FORTRAN source code for the simulation on the approximate procedure are attached in Appendix A. The FORTRAN code for the simultaneous comparison simulation of the approximate procedure and the nonparametric procedure are attached in Appendix C.

3. Analysis of simulation results

The simulation results are tabulated in Tables 3 through 5. The results shows that $\hat{R}_{1000(1-\alpha),L(1-\alpha)}$ is very close to the true reliability for every case of simulation, so that we have developed a very exact procedure. The procedure is more nearly exact for large sample sizes.

Table 3. ANALYSIS OF 80% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR EQUAL VARIANCES CASE WITH EQN. 2.12

R	σ	n	m	$\hat{R}_{1000(1-\alpha), L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	8	8	.8989	.8010	.0840	.0089
		8	30	.9003	.7980	.0628	.0056
		20	30	.9019	.7880	.0449	.0028
	20.0	8	8	.8989	.8010	.0840	.0089
		8	30	.9003	.7980	.0628	.0056
		20	30	.9019	.7880	.0449	.0028
.950	1.0	8	8	.9498	.8010	.0637	.0052
		8	30	.9511	.7910	.0450	.0030
		20	30	.9516	.7850	.0321	.0015
	20.0	8	8	.9499	.8010	.0636	.0052
		8	30	.9511	.7910	.0450	.0030
		20	30	.9517	.7840	.0320	.0015
.990	1.0	8	8	.9901	.7970	.0302	.0014
		8	30	.9906	.7830	.0185	.0006
		20	30	.9904	.7890	.0130	.0003
	20.0	8	8	.9901	.7970	.0302	.0014
		8	30	.9906	.7830	.0185	.0006
		20	30	.9904	.7890	.0130	.0003

Table 4. ANALYSIS OF 90% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR EQUAL VARIANCES CASE WITH EQN. 2.12

R	σ	n	m	$\hat{R}_{1000(1-\alpha),L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	8	8	.8944	.9060	.1390	.0111
		8	30	.9050	.8880	.0989	.0068
		20	30	.9034	.8880	.0703	.0033
	20.0	8	8	.8944	.9060	.1390	.0111
		8	30	.9050	.8880	.0989	.0068
		20	30	.9034	.8880	.0703	.0033
.950	1.0	8	8	.9467	.9090	.1081	.0074
		8	30	.9529	.8860	.0720	.0040
		20	30	.9517	.8890	.0507	.0019
	20.0	8	8	.9467	.9090	.1080	.0074
		8	30	.9530	.8860	.0719	.0040
		20	30	.9517	.8890	.0507	.0019
.990	1.0	8	8	.9886	.9110	.0549	.0025
		8	30	.9909	.8810	.0308	.0009
		20	30	.9904	.8950	.0211	.0004
	20.0	8	8	.9887	.9110	.0549	.0025
		8	30	.9909	.8810	.0308	.0009
		20	30	.9904	.8950	.0211	.0004

Table 5. ANALYSIS OF 95% CONFIDENCE LIMIT APPROXIMATION OF $R = P(X > Y)$ FOR EQUAL VARIANCES CASE WITH EQN. 2.12

R	σ	n	m	$\hat{R}_{1000(1-\alpha), L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	8	8	.8884	.9600	.1942	.0128
		8	30	.8987	.9520	.1330	.0079
		20	30	.9015	.9470	.0937	.0038
	20.0	8	8	.8884	.9600	.1942	.0128
		8	30	.8987	.9520	.1330	.0079
		20	30	.9015	.9470	.0937	.0038
.950	1.0	8	8	.9406	.9620	.1559	.0093
		8	30	.9489	.9510	.0987	.0049
		20	30	.9500	.9500	.0686	.0022
	20.0	8	8	.9406	.9620	.1558	.0093
		8	30	.9490	.9510	.0986	.0049
		20	30	.9500	.9490	.0686	.0022
.990	1.0	8	8	.9865	.9620	.0857	.0040
		8	30	.9900	.9500	.0444	.0014
		20	30	.9906	.9450	.0297	.0006
	20.0	8	8	.9865	.9620	.0857	.0040
		8	30	.9900	.9500	.0444	.0014
		20	30	.9906	.9450	.0297	.0006

III. NONPARAMETRIC DISTRIBUTION FREE LOWER CONFIDENCE BOUND PROCEDURE

The distribution-free confidence bound procedure suggested by Govindarajulu [Ref. 4: pp. 229-238], may be applied to our problem. However we should note that the Govindarajulu procedure requires large sample sizes and the true reliability values removed from 0 and 1. For comparative purposes only, we evaluated his procedure using the same computer simulation and analysis methods that were performed on our developed procedure. The same data was used to evaluate both procedures with emphases on $\hat{R}_{100(1-\alpha),L(1-\alpha)}$ and 'mean error'. The results are displayed in Tables 6 and 7.

Table 6. COMPARISON OF DIFFERENT PROCEDURES ON 90% LOWER CONFIDENCE LIMIT ESTIMATION OF R = P(X > Y), EQUAL VARIANCES CASE.

R	σ	n	m	Approximate Estimation		Nonparametric Bound	
				$\hat{R}_{1000(1-\alpha), I(1-\alpha)}$	Mean Error	$\hat{R}_{1000(1-\alpha), I(1-\alpha)}$	Mean Error
.900	1.0	25	30	.8998	.0650	.8225	.1289
		50	70	.8969	.0419	.8417	.0915
		90	90	.9000	.0322	.8602	.0676
	20.0	25	30	.8998	.0650	.8225	.1289
		50	70	.8970	.0418	.8417	.0915
		90	90	.9000	.0321	.8602	.0676
.950	1.0	25	30	.9488	.0466	.8545	.1289
		50	70	.9484	.0290	.8805	.0913
		90	90	.9499	.0220	.8997	.0677
	20.0	25	30	.9488	.0465	.8545	.1288
		50	70	.9485	.0288	.8805	.0912
		90	90	.9499	.0219	.8999	.0677
.990	1.0	25	30	.9894	.0190	.8718	.1286
		50	70	.9896	.0107	.9068	.0908
		90	90	.9900	.0079	.9284	.0676
	20.0	25	30	.9895	.0190	.8718	.1286
		50	70	.9896	.0107	.9068	.0908
		90	90	.9900	.0079	.9284	.0676

Table 7. COMPARISON OF DIFFERENT PROCEDURES ON 95% LOWER CONFIDENCE LIMIT ESTIMATION OF R = P(X > Y), EQUAL VARIANCES CASE.

R	σ	n	m	Approximate Estimation		Nonparametric Bound	
				$\hat{R}_{1000(1-\alpha), L(1-\alpha)}$	Mean Error	$\hat{R}_{1000(1-\alpha), L(1-\alpha)}$	Mean Error
.900	1.0	25	30	.8989	.0866	.7955	.1653
		50	70	.8976	.0549	.8245	.1172
		90	90	.8994	.0422	.8466	.0868
	20.0	25	30	.8989	.0866	.7955	.1653
		50	70	.8977	.0548	.8245	.1172
		90	90	.8995	.0422	.8466	.0868
.950	1.0	25	30	.9494	.0630	.8235	.1652
		50	70	.9477	.0384	.8597	.1170
		90	90	.9490	.0291	.8850	.0869
	20.0	25	30	.9495	.0629	.8235	.1652
		50	70	.9478	.0382	.8597	.1169
		90	90	.9491	.0290	.8850	.0868
.990	1.0	25	30	.9901	.0267	.8355	.1650
		50	70	.9895	.0146	.8820	.1165
		90	90	.9897	.0107	.9102	.0868
	20.0	25	30	.9901	.0266	.8355	.1649
		50	70	.9895	.0145	.8820	.1165
		90	90	.9897	.0106	.9102	.0867

IV. APPROXIMATE INTERVAL ESTIMATION PROCEDURE FOR RELIABILITY R = P(X > Y) — UNEQUAL VARIANCES CASE

A. LOWER CONFIDENCE LIMIT PROCEDURE

Let X denote component strength where $X \sim N(\mu_X, \sigma_X^2)$ Let Y denote stress applied to the component where $Y \sim N(\mu_Y, \sigma_Y^2)$. Then

$$X - Y \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

The component reliability is defined as follows:

$$\begin{aligned} R &= P[X > Y] \\ &= P\left[\frac{X - Y - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} > -\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \right] \\ &= \Phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \right), \end{aligned} \tag{4.1}$$

where Φ is the standard normal cumulative distribution function. Let

$$\delta = \frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}, \tag{4.2}$$

then $R = \Phi(\delta)$.

A consistent estimator of δ is

$$g = \frac{\bar{X} - \bar{Y}}{\sqrt{S_X^2 + S_Y^2}}, \tag{4.3}$$

where S_X^2 and S_Y^2 denotes sample variances and $\bar{X}$ and $\bar{Y}$ are the respective sample means.

For an observed value of $\frac{\bar{x}-\bar{y}}{\sqrt{s_X^2+s_Y^2}}$, the lower $100(1-\alpha)$ % confidence limit $\hat{\delta}_L$, for δ, using the general confidence interval method, is the value of δ in Equation (4.2) such that

$$1-\alpha = P\left[\frac{\bar{X}-\bar{Y}}{\sqrt{S_X^2+S_Y^2}} \le \frac{\bar{x}-\bar{y}}{\sqrt{s_X^2+s_Y^2}}\right]. \tag{4.4}$$

The probability statement in (4.4) cannot be reduced to an equivalent statement about a random variable whose distribution has been tabulated. This problem is similar to the Behrens-Fisher problem of finding a confidence interval for $\mu_X-\mu_Y$ when both variances σ_X^2 and σ_Y^2 are unknown and unequal. B. L. Welch [Ref. 10: pp. 28-35] has proposed an approximate confidence interval for $\mu_X-\mu_Y$ in this case using the statistic $\frac{\bar{X}-\bar{Y}}{\sqrt{S_X^2/n+S_Y^2/m}}$. Welch approximated the distribution of this statistic with a Student's t distribution with v degrees of freedom. The degrees of freedom chosen by Welch is given by

$$v = \frac{(S_X^2/n+S_Y^2/m)^2}{\frac{S_X^2/n^2}{n-1}+\frac{S_Y^2/m^2}{m-1}}.$$

The Welch statistic is not useful in our problem, because we need to use a statistic that is a consistent estimator for $\frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}$.

In particular we choose to use the statistic $g=\frac{\bar{X}-\bar{Y}}{\sqrt{S_X^2+S_Y^2}}$. The mean and variance of this statistic are approximated and the distribution of $\frac{g-\mathrm{E}[g]}{\hat{\sigma}_g}$ is approximated with a Student's t distribution. The desired confidence interval is constructed using the approximated Student's t distribution. The analysis in the following paragraph serves only as a means to find a plausible expression for the degrees of freedom of the approximated Student's distribution.

We begin as in equation (4.4), where $\frac{(\bar{x}-\bar{y})}{\sqrt{s_X^2+s_Y^2}}$ denotes a value constructed from observed data. The upper case version of the same expression denotes a random variable.

$$
\begin{aligned}
1-\alpha &= P\left[\frac{\bar{X}-\bar{Y}}{\sqrt{S_X^2+S_Y^2}} \le \frac{\bar{x}-\bar{y}}{\sqrt{s_X^2+s_Y^2}}\right] \\
&= P\left[\frac{\dfrac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)+(\mu_X-\mu_Y)}{\sqrt{\sigma_X^2/n+\sigma_Y^2/m}}}{\dfrac{\sqrt{S_X^2+S_Y^2}}{\sqrt{\sigma_X^2/n+\sigma_Y^2/m}}} \le \frac{\bar{x}-\bar{y}}{\sqrt{s_X^2/n+s_Y^2/m}}\,\frac{\sqrt{s_X^2/n+s_Y^2/m}}{\sqrt{s_X^2+s_Y^2}}\right] \\
&= P\left[\frac{Z+\dfrac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2/n+\sigma_Y^2/m}}}{\sqrt{\dfrac{S_X^2+S_Y^2}{\sigma_X^2/n+\sigma_Y^2/n}}\sqrt{\dfrac{s_X^2/n+s_Y^2/m}{s_X^2+s_Y^2}}} \le \frac{\bar{x}-\bar{y}}{\sqrt{s_X^2/n+s_Y^2/m}}\right].
\end{aligned}
\tag{4.5}
$$

where σ_X^2, σ_Y^2, s_X^2, and s_Y^2 denote constants. The last probability in Equation (4.5) suggests a noncentral t distribution. Consequently the random variable

$$\sqrt{\frac{(s_X^2/n+s_Y^2/m)(S_X^2+S_Y^2)}{(\sigma_X^2/n+\sigma_Y^2/m)(s_X^2+s_Y^2)}}$$

in the denominator should be a random variable of the form $\sqrt{\frac{\chi_\nu^2}{\nu}}$. That is,

$$\frac{s_X^2/n+s_Y^2/m}{\sigma_X^2/n+\sigma_Y^2/m}\,\frac{S_X^2+S_Y^2}{s_x^2+s_Y^2}\,\nu = \chi_\nu^2.$$

Since $\mathrm{Var}(\chi_\nu^2)=2\nu$, $\mathrm{Var}(S_X^2)=\frac{2\sigma_X^4}{n-1}$, and $\mathrm{Var}(S_Y^2)=\frac{2\sigma_Y^4}{m-1}$,

$$2\nu = \nu^2\left(\frac{s_X^2/n+s_Y^2/m}{s_X^2+s_Y^2}\right)^2\frac{1}{(\sigma_X^2/n+\sigma_Y^2/m)^2}\left(\frac{2\sigma_X^4}{n-1}+\frac{2\sigma_Y^4}{m-1}\right).$$

Thus,

$$
\begin{aligned}
v &= \left(\frac{s_X^2 + s_Y^2}{s_X^2/n + s_Y^2/m} \right)^2 \frac{(\sigma_X^2/n + \sigma_Y^2/m)^2}{\sigma_X^4/(n-1) + \sigma_Y^4/(m-1)} \\
&\doteq \left(\frac{s_X^2 + s_Y^2}{s_X^2/n + s_Y^2/m} \right)^2 \frac{(s_X^2/n + s_Y^2/m)^2}{s_X^4/(n-1) + s_Y^4/(m-1)} \\
&= \frac{(s_X^2 + s_Y^2)^2}{s_X^4/(n-1) + s_Y^4/(m-1)} \qquad (4.6)
\end{aligned}
$$

In summary, if we fit a t distribution to the distribution of $\frac{g - \mathrm{E}[g]}{\hat{\sigma}_g}$, the expression in the Equation (4.6) might be a plausible random value for the degrees of freedom. The results of computer simulations will indicate the accuracy of this approximation.

In order to formulate the lower confidence interval statement for $\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}$, we first find the mean, $\mathrm{E}[g] = \mu_g$, and variance, $\mathrm{Var}[g] = \sigma_g^2$, for $g = \frac{\overline{X} - \overline{Y}}{\sqrt{S_X^2 + S_Y^2}}$. We then approximate the distribution of $\frac{g - \mu_g}{\hat{\sigma}_g}$ with a central Student's t distribution with v degrees of freedom, where v is given in equation (4.6). The lower confidence limit, $\hat{\mu}_{g_{L,1-\alpha}}$ for μ_g will yield the corresponding lower confidence limit for reliability

$$
\hat{R}_{L,1-\alpha} = \Phi(\hat{\mu}_{g_{L,1-\alpha}}).
$$

We now proceed to find μ_g and σ_g^2. The Taylor expansion of g at $(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$, using only first order derivatives is given by:

$$
\begin{aligned}
g(\overline{X} - \overline{Y},\ S_X^2 + S_Y^2) &= g(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2) + [\overline{X} - \overline{Y} - (\mu_X - \mu_Y)] \frac{\partial g}{\partial(\overline{X} - \overline{Y})} \bigg|_{(\mu_X - \mu_Y,\ \sigma_X^2 - \sigma_Y^2)} \\
&\quad + [S_X^2 + S_Y^2 - (\sigma_X^2 + \sigma_Y^2)] \frac{\partial g}{\partial(S_X^2 + S_Y^2)} \bigg|_{(\mu_X - \mu_Y,\ \sigma_X^2 - \sigma_Y^2)} + R_\varepsilon \qquad (4.7) \\
&= \frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}} + \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} - [S_X^2 + S_Y^2 - (\sigma_X^2 + \sigma_Y^2)] \frac{\mu_X - \mu_Y}{2(\sigma_X^2 + \sigma_Y^2)^{3/2}} + R_\varepsilon
\end{aligned}
$$

where R_ε includes terms that converge to 0 at the same rate as $\max\left(\frac{1}{n}, \frac{1}{m}\right)$, as n and m become large.

The expected value and variance of g are

$$E[g] = \mu_g \doteq \frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}, \tag{4.8}$$

and

$$\begin{aligned} \mathrm{Var}[g] &= \sigma_g^2 \\ &\doteq \frac{1}{\sigma_X^2 + \sigma_Y^2} \mathrm{Var}(\overline{X} - \overline{Y}) + \frac{(\mu_X - \mu_Y)^2}{4(\sigma_X^2 + \sigma_Y^2)^{3/2}} \mathrm{Var}(\sigma_X^2 + \sigma_Y^2) \\ &= \frac{\sigma_X^2/n + \sigma_Y^2/m}{\sigma_X^2 + \sigma_Y^2} + \frac{(\mu_X - \mu_Y)^2}{4(\sigma_X^2 + \sigma_Y^2)^3} \left(\frac{2\sigma_X^4}{n-1} + \frac{2\sigma_Y^4}{m-1} \right) \\ &= \frac{\sigma_X^2/n + \sigma_Y^2/m}{\sigma_X^2 + \sigma_Y^2} + \frac{(\mu_X - \mu_Y)^2}{2(\sigma_X^2 + \sigma_Y^2)^3} \left(\frac{\sigma_X^4}{n-1} + \frac{\sigma_Y^4}{m-1} \right). \end{aligned} \tag{4.9}$$

An estimator for σ_g is

$$\hat{\sigma}_g = \left[\frac{S_X^2/n + S_Y^2/m}{S_X^2 + S_Y^2} + \frac{(\overline{X} - \overline{Y})^2}{2(S_X^2 + S_Y^2)^3} \left(\frac{S_X^4}{n-1} + \frac{S_Y^4}{m-1} \right) \right]^{1/2}. \tag{4.10}$$

Then an approximate confidence limit for μ_g is the follows:

$$\begin{aligned} \hat{\mu}_{g_{L,1-\alpha}} &= g - \hat{\sigma}_g t_{1-\alpha,\, v} \\ &= \hat{\delta}_{L,\,1-\alpha}. \end{aligned} \tag{4.11}$$

The corresponding $100(1-\alpha)$ % lower confidence limit, $\hat{R}_{L,1-\alpha}$, for component reliability is

$$\hat{R}_{L,\,1-\alpha} = \Phi(\hat{\delta}_{L,\,1-\alpha}). \tag{4.12}$$

1. Example

Let x_i denote the strength of solid missile motor chambers that were pressurized until they burst, $i = 1, 2, .., 25$. Let y_j denote the maximum pressure observed on 30 solid missile motors that use this type of chamber. X and Y are assumed to have normal distributions with unequal variances. The coded X and Y data are:

X: 292.65 301.86 297.86 312.67 300.61 292.18 297.55

284.64 297.93 295.22 315.49 294.10 299.85 292.24

312.25 298.87 295.41 292.41 287.71 298.57 300.34

303.28 309.52 283.32 308.52

Y: 242.52 261.00 229.37 261.82 248.36 174.04 192.50

265.64 199.70 225.52 237.43 285.66 199.20 203.65

274.93 302.13 302.85 233.59 275.95 220.46 318.34

269.86 200.10 318.19 216.69 281.78 301.20 348.56

234.15 234.45

The corresponding 90% lower confidence limit for component reliability, $R = P(X > Y)$ is computed as follows:

$$n = 25, \quad \bar{x} = 298.60, \quad s_X^2 = 70.25$$

$$m = 30, \quad \bar{y} = 251.99, \quad s_Y^2 = 1873.79$$

$$v \doteq \frac{(s_X^2 + s_Y^2)^2}{s_X^4/(n-1) + s_Y^4/(m-1)} = 31$$

$$\hat{\sigma}_g = \left[\frac{s_X^2/n + s_Y^2/m}{s_X^2 + s_Y^2} + \frac{(\bar{x} - \bar{y})^2}{2(s_X^2 + s_Y^2)^3} \left(\frac{s_X^4}{n-1} + \frac{s_Y^4}{m-1} \right) \right]^{1/2} = 0.23$$

$$g = \frac{\bar{x} - \bar{y}}{\sqrt{s_X^2 + s_Y^2}} = 1.06$$

$$\hat{\delta}_{L,\,0.9} = g - \hat{\sigma}_g t_{0.9,\,31} = 0.76$$

$$\hat{R}_{L,\,0.9} = \Phi\left(\hat{\delta}_{L,\,0.9}\right) = 0.78$$

The corresponding 90% Govindarajulu nonparametric confidence limit is 0.69. This is not surprising because our procedure uses more information about the distributions of X and Y. Intervals estimated by nonparametric procedures are usually wider than those estimated by parametric procedures. The amount of the difference is somewhat surprising for these sample sizes of 25 and 30.

B. ACCURACY OF THE PROCEDURE

1. Computer simulation

a. Simulation procedure

Computer simulations were used to determine its accuracy for specific sets values of n, m, μ_X, μ_Y, σ_X^2, and σ_Y^2.

Parameters are chosen in a way to cover practical conditions. The sets of parameters are as follows:

- $\mu_X > \mu_Y$
- Reliability, R : 0.90, 0.95, 0.99
- Standard Deviations

  σ_X: 1.0, 10.0
  σ_Y: 2.0, 40.0

- Sample sizes, (n. m): (10, 20); (25, 35); (75, 50)
- Confidence level, α : 0.05, 0.10, 0.20 .

Actual sets of parameters are tabulated in Appendix E.

Simulation procedure is summarized as follows:

1) Generate n random normal variates X, $X \sim N(\mu_X, \sigma_X^2)$; m random normal variates Y, $Y \sim N(\mu_Y, \sigma_Y^2)$.
2) Compute $\overline{X}$, $\overline{Y}$, S_X^2. S_Y^2.
3) Compute $\hat{\sigma}_g, \hat{\delta}_{L,1-\alpha}$.
4) Compute $\hat{R}_{L,1-\alpha} = \Phi(\hat{\delta}_{L,1-\alpha})$, for α = 0.05, 0.10, 0.20.
5) Repeat steps 1 through 4, for 1000 times. Then order the $\hat{R}_{L,1-\alpha}$ to get $\hat{R}_{(1),L(1-\alpha)}$, $\hat{R}_{(2),L(1-\alpha)}$, ..., $\hat{R}_{100\%(1-\alpha),L(1-\alpha)}$, ..., $\hat{R}_{(1000),L(1-\alpha)}$, from the smallest to the largest.

6) Print $\hat{R}_{1000(1-\alpha),\,L(1-\alpha)}$.

7) Print $\max_i\{\hat{R}_{(i),\,L(1-\alpha)}:\ \hat{R}_{(i),\,L(1-\alpha)} \leq R\}$,the actual confidence level of the approximation will be $\frac{i}{1000}$.

8) Compute and print the sample mean and variance of the estimation error.

b. Simulation language

The simulations were conducted on the N.P.S. mainframe IBM 3033 computer. The programming language used for the simulations is VS FORTRAN 2. The random number generator LLRANII was used to generate normal variates. The IMSL statistics function TIN was used to compute the percentile of Student's t distribution, and ANORDF was used to compute the probability of standard normal distribution. The FORTRAN source code for the simulation of the developed approximate procedure are attached in Appendix D. The FORTRAN code for simultaneous comparison between the approximate procedure and nonparametric procedure are attached in Appendix F.

2. Analysis of simulation results

The simulation results are tabulated in Tables 8 through 10. The results show the developed approximate procedure is quite accurate, because for every case the $100(1-\alpha)^{\text{th}}$ percentile point of $\hat{R}_{1-\alpha}$ are all very close to the true reliability. Also, the 'mean error from R' and 'Variance of error' reduce rapidly with the increases of sample size. Comparison simulations were run for the approximate procedure versus the Govindarajulu nonparametric procedure. Results are tabulated in table 11 and 12.

Table 8. ANALYSIS OF 80% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR UNEQUAL VARIANCES CASE

R	σ_X	σ_Y	n	m	$\hat{R}_{1000(1-\alpha),\, L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	2.0	10	20	.9008	.7960	.0552	.0044
			25	35	.8996	.8030	.0388	.0021
			75	50	.8997	.8010	.0298	.0011
	10.0	40.0	10	20	.9019	.7910	.0530	.0042
			25	35	.9012	.7900	.0392	.0022
			75	50	.9000	.8000	.0317	.0013
.950	1.0	2.0	10	20	.9500	.8000	.0409	.0024
			25	35	.9502	.7980	.0278	.0011
			75	50	.9496	.8080	.0211	.0006
	10.0	40.0	10	20	.9513	.7840	.0401	.0024
			25	35	.9514	.7820	.0287	.0012
			75	50	.9487	.8100	.0229	.0007
.990	1.0	2.0	10	20	.9902	.7930	.0180	.0005
			25	35	.9903	.7890	.0112	.0002
			75	50	.9898	.8110	.0082	.0001
	10.0	40.0	10	20	.9906	.7770	.0186	.0006
			25	35	.9907	.7750	.0122	.0002
			75	50	.9899	.8030	.0093	.0001

Table 9. ANALYSIS OF 90% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR UNEQUAL VARIANCES CASE

R	σ_X	σ_Y	n	m	$\hat{R}_{1000(1-\alpha), L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	2.0	10	20	.9008	.8970	.0893	.0052
			25	35	.8993	.9010	.0611	.0024
			75	50	.8990	.9070	.0465	.0013
	10.0	40.0	10	20	.9022	.8890	.0869	.0050
			25	35	.9005	.8970	.0624	.0025
			75	50	.8995	.9020	.0499	.0015
.950	1.0	2.0	10	20	.9497	.9010	.0671	.0032
			25	35	.9502	.8990	.0443	.0014
			75	50	.9493	.9030	.0332	.0007
	10.0	40.0	10	20	.9510	.8960	.0667	.0032
			25	35	.9494	.9020	.0462	.0015
			75	50	.9494	.9040	.0364	.0008
.990	1.0	2.0	10	20	.9899	.9060	.0308	.0009
			25	35	.9899	.9010	.0184	.0003
			75	50	.9899	.9050	.0132	.0001
	10.0	40.0	10	20	.9901	.8980	.0321	.0009
			25	35	.9904	.8930	.0203	.0004
			75	50	.9898	.9060	.0152	.0002

Table 10. ANALYSIS OF 95% CONFIDENCE LIMIT APPROXIMATION OF R = P(X > Y) FOR UNEQUAL VARIANCES CASE

R	σ_X	σ_Y	n	m	$\hat{R}_{100(1-\alpha), L(1-\alpha)}$	True confidence level	Mean error from R	Variance of error
.900	1.0	2.0	10	20	.8958	.9560	.1218	.0059
			25	35	.8988	.9530	.0816	.0027
			75	50	.8998	.9500	.0615	.0014
	10.0	40.0	10	20	.9009	.9470	.1196	.0057
			25	35	.8980	.9540	.0838	.0028
			75	50	.8992	.9520	.0664	.0017
.950	1.0	2.0	10	20	.9461	.9550	.0934	.0039
			25	35	.9482	.9590	.0600	.0016
			75	50	.9500	.9480	.0445	.0008
	10.0	40.0	10	20	.9517	.9450	.0938	.0039
			25	35	.9470	.9550	.0632	.0018
			75	50	.9500	.9500	.0492	.0010
.990	1.0	2.0	10	20	.9888	.9550	.0452	.0013
			25	35	.9891	.9610	.0259	.0004
			75	50	.9899	.9520	.0182	.0002
	10.0	40.0	10	20	.9903	.9420	.0478	.0014
			25	35	.9892	.9550	.0288	.0005
			75	50	.9900	.9490	.0212	.0002

Table 11. COMPARISON OF DIFFERENT PROCEDURES ON 90% LOWER CONFIDENCE LIMIT ESTIMATION OF R = P(X > Y), UNEQUAL VARIANCES CASE

R	σ_X	σ_Y	n	m	Approximate Estimation		Nonparametric Bound	
					$\hat{R}_{1000(1-\gamma),\,L(1-\gamma)}$	Mean error	$\hat{R}_{1000(1-\gamma),\,L(1-\gamma)}$	Mean error
.900	1.0	2.0	10	15	.9013	.1004	.7707	.2018
			70	35	.9003	.0564	.8370	.1098
			90	90	.9009	.0333	.8623	.0673
	10.0	40.0	10	15	.9056	.1010	.7840	.2017
			70	35	.9010	.0613	.8452	.1094
			90	90	.9001	.0349	.8684	.0672
.950	1.0	2.0	10	15	.9523	.0767	.7974	.2015
			70	35	.9506	.0410	.8729	.1091
			90	90	.9507	.0233	.9027	.0675
	10.0	40.0	10	15	.9550	.0790	.7974	.2014
			70	35	.9505	.0456	.8807	.1090
			90	90	.9497	.0250	.9067	.0672
.990	1.0	2.0	10	15	.9915	.0370	.7974	.2021
			70	35	.9906	.0172	.8913	.1088
			90	90	.9901	.0088	.9301	.0676
	10.0	40.0	10	15	.9921	.0402	.7974	.2021
			70	35	.9904	.0200	.8917	.1087
			90	90	.9901	.0098	.9320	.0675

Table 12. COMPARISON OF DIFFERENT PROCEDURES ON 95% LOWER CONFIDENCE LIMIT ESTIMATION OF R = P(X > Y), UNEQUAL VARIANCES CASE

R	σ_X	σ_Y	n	m	Approximate Estimation		Nonparametric Bound	
					$\hat{R}_{1000(1-\alpha), l(1-\alpha)}$	Mean Error	$\hat{R}_{1000(1-\alpha), l(1-\alpha)}$	Mean Error
.900	1.0	2.0	10	15	.8988	.1388	.7266	.2593
			70	35	.8977	.0750	.8194	.1405
			90	90	.9014	.0439	.8529	.0865
	10.0	40.0	10	15	.8975	.1414	.7399	.2592
			70	35	.8950	.0823	.8275	.1401
			90	90	.9005	.0463	.8575	.0863
.950	1.0	2.0	10	15	.9506	.1086	.7399	.2590
			70	35	.9477	.0553	.8483	.1398
			90	90	.9503	.0310	.8890	.0866
	10.0	40.0	10	15	.9503	.1135	.7399	.2589
			70	35	.9468	.0622	.8557	.1397
			90	90	.9509	.0334	.8927	.0864
.990	1.0	2.0	10	15	.9896	.0557	.7399	.2595
			70	35	.9898	.0240	.8610	.1395
			90	90	.9903	.0119	.9117	.0867
	10.0	40.0	10	15	.9907	.0617	.7399	.2595
			70	35	.9893	.0285	.8610	.1395
			90	90	.9901	.0134	.9132	.0867

V. CONCLUSIONS AND RECOMMENDATIONS

The lower confidence limit estimation procedures developed in this thesis for equal variances case as well as for unequal variances cases, are very accurate. These procedures are simple to evaluate and require only the use of central Student's t tables, in contrast to the existing parametric procedures of this type which require extensive use of noncentral Student's t tables.

Although these procedures are developed for lower confidence limits, upper or two-sided confidence limits for the reliability are readily obtained.

APPENDIX A. FORTRAN CODE FOR INTERVAL ESTIMATION PROCEDURE - NORMAL EQUAL VARIANCES CASE

```
      PROGRAM EQSIGM
***********************************************************************
*                                                                     *
*    THIS PROGRAM IS TO VALIDATE THE LOWER CONFIDENCE BOUND APPROXI-  *
*    MATION PROCEDURE FOR P( X > Y ), WHERE X, Y ARE NORMALLY DIS-    *
*    TRIBUTED WITH UNKNOWN MEANS AND A UNKNOWN BUT EQUAL VARIANCE.    *
*                                                                     *
*    VARIABLES DESCRIPTION:                                           *
*                                                                     *
*         ALPHA    -  NOMINAL CONFIDENCE LEVEL                        *
*         ANORDF   -  IMSL FUNCTION FOR NORMAL PROBABILITY            *
*         CASE     -  NUMBER OF TEST PARAMETER SETS                   *
*         ER       -  ERROR BETWEEN LIMIT AND TRUE RELIABILITY        *
*         ERBAR    -  AVERAGE OF ER                                   *
*         ERSSQ    -  SUM OF SQUARES OF ER                            *
*         ERSUM    -  SUM OF ER                                       *
*         ERSV     -  SAMPLE VARIANCE OF ER                           *
*         CLOSE    -  INDEX OF THE CLOSEST ESTIMATE                   *
*         DELTA    -  NONCENTRALITY OF T DISTRIBUTION                 *
*         DF       -  DEGREE OF FREEDOM OF T DISTRIBUTION             *
*         K        -  STATISTIC TO ESTIMATE DELTA                     *
*         LNORM    -  RANDOM NUMBER GENERATOR FOR NORMAL VARIATES     *
*         M        -  SAMPLE SIZE OF Y RANDOM VARIABLE                *
*         MUX      -  POPULATION MEAN OF X RANDOM VARIABLE            *
*         MUY      -  POPULATION MEAN OF Y RANDOM VARIABLE            *
*         N        -  SAMPLE SIZE OF X RANDOM VARIABLE                *
*         R        -  REAL RELIABILITY                                *
*         REP      -  REPETITION OF SIMULATION                        *
*         RLHAT    -  LOWER CONFIDENCE LIMIT OF RELIABILITY           *
*         SIGMAX   -  POPULATION STANDARD DEVIATION OF X              *
*         SIGMAY   -  POPULATION STANDARD DEVIATION OF Y              *
*         SP       -  POOLED SAMPLE VARIANCE OF X AND Y               *
*         SUMSQX   -  SUM OF SQUARES OF X                             *
*         SUMSQY   -  SUM OF SQUARES OF Y                             *
*         SUMX     -  SUM OF X                                        *
*         SUMY     -  SUM OF Y                                        *
*         TIN      -  IMSL FUNCTION TO COMPUTE PERCENTILE OF T DIST.  *
*         XBAR     -  AVERAGE OF X                                    *
*         YBAR     -  AVERAGE OF Y                                    *
*                                                                     *
*    REQUIRED EXTERNAL FUNCTIONS: LNORM, TIN, ANORDF                  *
*                                                                     *
***********************************************************************
*
      INTEGER REP, CASE
      REAL ALPHA, TWO
      PARAMETER(ALPHA=0.05)
      PARAMETER(TWO=2.0)
```

```
      PARAMETER(REP=1000)
      PARAMETER(CASE=18)
*
      INTEGER I, I1, J, U, V, XSEED, YSEED, N, M, CLOSE
      REAL MUX, MUY, XBAR, YBAR, SIGMAX, SIGMAY, DF, R, RLHAT(REP),
     +     K, DELTA, XX(100), YY(100), X, Y, SUMSQX, SUMSQY, SUMX,
     +     SUMY, TIN, ANORDF, RN, RM, DIFF, SP, TEMP, ER, ERSUM,
     +     ERSSQ, ERBAR, ERSV
*
      CALL EXCMS('FILEDEF 12 DISK SETUP1 DATA A1')
      CALL EXCMS('FILEDEF 18 DISK OPT1 DATA A1')
*
      DO 2000 I=1, CASE
      READ (12,2200) XSEED, YSEED, MUX, MUY, SIGMAX, SIGMAY, N, M, R
      DF = REAL(N+M-2)
      RN = REAL(N)
      RM = REAL(M)
      ERSUM = 0.0
      ERSSQ = 0.0
      DO 1000 J=1, REP
         CALL LNORM(XSEED, XX, N, 2, 0)
         CALL LNORM(YSEED, YY, M, 2, 0)
         SUMSQX = 0.0
         SUMSQY = 0.0
         SUMX = 0.0
         SUMY = 0.0
*
*        < TRANSFORM X, Y TO DESIRED PROPERTIES >
*
         DO 200 U= 1, N
            X = XX(U) * SIGMAX + MUX
            SUMSQX= SUMSQX + X * X
            SUMX = SUMX + X
200      CONTINUE
         XBAR= SUMX / RN
         DO 300 V=1, M
            Y = YY(V) * SIGMAY + MUY
            SUMSQY = SUMSQY + Y * Y
            SUMY = SUMY + Y
300      CONTINUE
*
*        < COMPUTE CONFIDENCE LIMIT FOR RELIABILITY >
*
         YBAR = SUMY / RM
         SP = SQRT( (SUMSQX - RN*XBAR*XBAR + SUMSQY - RM*YBAR*YBAR)
     +             / DF)
         K = MAX( (XBAR - YBAR) / SP, 0.0 )
         DELTA = K - SQRT( (RN+RM)/(RN*RM) + K*K / (2.0*(RN+RM-2.0)))
     +             * TIN(1.0-ALPHA,DF)
         RLHAT(J) = ANORDF(DELTA/SQRT(TWO))
*
*        < COMPUTE THE MEAN AND VARIANCE OF ER >
*
         ER = R - RLHAT(J)
         ERSUM = ERSUM + ER
```

```
         ERSSQ = ERSSQ + ER * ER
1000  CONTINUE
         ERBAR = ERSUM / REAL(REP)
         ERSV = ( ERSSQ - REAL(REP) * ERBAR * ERBAR ) / REAL(REP-1)
*
*      < SORT CONFIDENCE LIMITS IN ASCENDING ORDER >
*
      DIFF = 2.0
      DO 1800 I1=1, REP
         DO 1500 J=I1+1, REP
            IF (RLHAT(J) .LT. RLHAT(I1))  THEN
               TEMP = RLHAT(I1)
               RLHAT(I1) = RLHAT(J)
               RLHAT(J) = TEMP
            ENDIF
1500     CONTINUE
*
*        < FIND THE CLOSEST CONFIDENCE LIMIT ESTIMATE >
*
         IF (((R-RLHAT(I1)) .GE.  0.1E-6) .AND.
     +   ((R-RLHAT(I1)) .LE.  DIFF)) THEN
            DIFF = R -  RLHAT(I1)
            CLOSE = I1
         ENDIF
1800  CONTINUE
      WRITE (18,2100) I, MUX, N, SIGMAX, MUY, M, SIGMAY, R,
     +                RLHAT(NINT(REAL(REP)*(1.0-ALPHA))), RLHAT(CLOSE),
     +                REAL(CLOSE)/1000.0, ERBAR, ERSV
2000  CONTINUE
2100  FORMAT('SIMULATION: ',I2,/,'MUX: ',F5.1,T16,'N: ',I2,
     +       T35,'SIGMAX: ',F4.1,/,'MUY: ',F7.3,T16,'M: ',I2,
     +       T35,'SIGMAY: ',F4.1,/,'TRUE R : ',F7.5,
     +       T35,'RLHAT: ',F7.5,/,'CLOSEST RLHAT: ',F7.5,
     +       T35,'TRUE CONFIDENCE LEVEL: ',F5.3,/,
     +       'MEAN ERROR WIDTH: ',F7.5,/,
     +       'VARIANCE OF ERROR: ',F7.5,///)
2200  FORMAT (I5,1X,I5,1X,F5.1,1X,F7.3,1X,F4.1,1X,F4.1,1X,I2,1X,I2,1X,
     +F5.3)
      STOP
      END
```

APPENDIX B. SIMULATION PARAMETER SETS FOR EQUAL VARIANCES CASE

FILE: SETUP1 DATA (FOR APPROXIMATE PROCEDURE)

X SEED	Y SEED	X MEAN	Y MEAN	X STDV	Y STDV	N	M	R	
16807	93943	10.0	8.187	1.0	1.0	8	8	.900	1
16807	93943	10.0	8.187	1.0	1.0	8	30	.900	2
16807	93943	10.0	8.187	1.0	1.0	20	30	.900	3
16807	93943	100.0	63.740	20.0	20.0	8	8	.900	4
16807	93943	100.0	63.740	20.0	20.0	8	30	.900	5
16807	93943	100.0	63.740	20.0	20.0	20	30	.900	6
16807	93943	10.0	7.674	1.0	1.0	8	8	.950	7
16807	93943	10.0	7.674	1.0	1.0	8	30	.950	8
16807	93943	10.0	7.674	1.0	1.0	20	30	.950	9
16807	93943	100.0	53.472	20.0	20.0	8	8	.950	10
16807	93943	100.0	53.472	20.0	20.0	8	30	.950	11
16807	93943	100.0	53.472	20.0	20.0	20	30	.950	12
16807	93943	10.0	6.711	1.0	1.0	8	8	.990	13
16807	93943	10.0	6.711	1.0	1.0	8	30	.990	14
16807	93943	10.0	6.711	1.0	1.0	20	30	.990	15
16807	93943	100.0	34.211	20.0	20.0	8	8	.990	16
16807	93943	100.0	34.211	20.0	20.0	8	30	.990	17
16807	93943	100.0	34.211	20.0	20.0	20	30	.990	18

FILE: SEQ4 DATA (FOR COMPARISON SIMULATIONS)

```
16807 93943  10.0    8.187  1.0   1.0 25 30   .900          1
16807 93943  10.0    8.187  1.0   1.0 50 70   .900          2
16807 93943  10.0    8.187  1.0   1.0 90 90   .900          3
16807 93943 100.0   63.740 20.0  20.0 25 30   .900          4
16807 93943 100.0   63.740 20.0  20.0 50 70   .900          5
16807 93943 100.0   63.740 20.0  20.0 90 90   .900          6
16807 93943  10.0    7.674  1.0   1.0 25 30   .950          7
16807 93943  10.0    7.674  1.0   1.0 50 70   .950          8
16807 93943  10.0    7.674  1.0   1.0 90 90   .950          9
16807 93943 100.0   53.472 20.0  20.0 25 30   .950          10
16807 93943 100.0   53.472 20.0  20.0 50 70   .950          11
16807 93943 100.0   53.472 20.0  20.0 90 90   .950          12
16807 93943  10.0    6.711  1.0   1.0 25 30   .990          13
16807 93943  10.0    6.711  1.0   1.0 50 70   .990          14
16807 93943  10.0    6.711  1.0   1.0 90 90   .990          15
16807 93943 100.0   34.211 20.0  20.0 25 30   .990          16
16807 93943 100.0   34.211 20.0  20.0 50 70   .990          17
16807 93943 100.0   34.211 20.0  20.0 90 90   .990          18
----- ----- -----   ------ ----  ---- -- --   ----
I5    I5    F5.1    F7.3   F4.1  F4.1 I2 I2   F5.3
X     Y     X       Y      X     Y    N  M    R
SEED  SEDD  MEAN    MEAN   STDV  STDV
```

APPENDIX C. FORTRAN CODE FOR SIMULTANEOUS COMPARISON SIMULATION OF APPROXIMATE PROCEDURE VS. NONPARAMETRIC PROCEDURE - EQUAL VARIANCES

```
      PROGRAM COMEQ
**************************************************************************
*    THIS PROGRAM IS TO COMPUTE THE LOWER CONFIDENCE LIMIT OF            *
*    R = P( X > Y ), WHERE X, Y ARE INDEPENDENTLY NORMALLY DIS-          *
*    TRIBUTED WITH UNKNOWN MEANS AND A UNKNOWN BUT EQUAL VARIAN-         *
*    CES WITH APPROXIMATE PROCEDURE AND WITH NONPARAMETRIC PRO-          *
*    CEDURE                                                              *
*                                                                        *
*    VARIABLES DESCRIPTION:                                              *
*                                                                        *
*         ALPHA   -  NOMINAL CONFIDENCE LEVEL                            *
*         ANORDF  -  IMSL FUNCTION FOR NORMAL PROBABILITY                *
*         ANORIN  -  IMSL FUNCTION FOR INVERSE NORMAL CDF                *
*         CASE    -  NUMBER OF TEST PARAMETER SETS                       *
*         BIGU    -  MANN-WHITNEY STATISTIC                              *
*         CLOSE   -  INDEX OF THE CLOSEST ESTIMAT (APPROXI PROCEDURE) *
*         CLOSEN  -  INDEX OF THE CLOSEST ESTIMAT (NONPARA PROCEDURE) *
*         DELTA   -  NONCENTRALITY OF T DISTRIBUTION                     *
*         DF      -  DEGREE OF FREEDOM OF T DISTRIBUTION                 *
*         EPS     -  WIDTH OF THE CONFIDENCE BOUND                       *
*         EROR    -  EROR OF ESTIMATION  (APPROXI PROCEDURE)             *
*         ERORN   -  EROR OF ESTIMATION  (NONPARA PROCEDURE)             *
*         ERBAR   -  MEAN OF EROR (APPROXI PROCEDURE)                    *
*         ERBARN  -  MEAN OF EROR (NONPARA PROCEDURE)                    *
*         ERSSQ   -  SUM OF SQUARE OF EROR (APPROXI PROCEDURE)           *
*         ERSSQN  -  SUM OF SQUARE OF EROR (NONPARA PROCEDURE)           *
*         ERSUM   -  SUM OF EROR (APPROXI PROCEDURE)                     *
*         ERSUMN  -  SUM OF EROR (NONPARA PROCEDURE)                     *
*         ERSV    -  SAMPLE VARIANCE OF EROR (APPROXI PROCEDURE)         *
*         ERSVN   -  SAMPLE VARIANCE OF EROR (NONPARA PROCEDURE)         *
*         K       -  STATISTIC TO ESTIMATE DELTA                         *
*         LNORM   -  RANDOM NUMBER GENERATOR FOR NORMAL VARIATES         *
*         M       -  SAMPLE SIZE OF Y RANDOM VARIABLE                    *
*         MUX     -  POPULATION MEAN OF X RANDOM VARIABLE                *
*         MUY     -  POPULATION MEAN OF Y RANDOM VARIABLE                *
*         N       -  SAMPLE SIZE OF X RANDOM VARIABLE                    *
*         NU      -  THE SMALLER OF SAMPLE SIZES                         *
*         R       -  REAL RELIABILITY                                    *
*         RB      -  CONFIDENCE BOUND OF THE RELIABILITY                 *
*         REP     -  REPETITION OF SIMULATION                            *
*         RLHAT   -  LOWER CONFIDENCE LIMIT OF RELIABILITY               *
*         RTILD   -  POINT ESTIMATOR OF THE RELIABILITY                  *
*         SIGMAX  -  POPULATION STANDARD DEVIATION OF X                  *
*         SIGMAY  -  POPULATION STANDARD DEVIATION OF Y                  *
*         SP      -  POOLED SAMPLE VARIANCE OF X AND Y                   *
*         SUMSQX  -  SUM OF SQUARES OF X                                 *
```

```
*         SUMSQY    -  SUM OF SQUARES OF Y                                 *
*         SUMX      -  SUM OF X                                            *
*         SUMY      -  SUM OF Y                                            *
*         TIN       -  IMSL FUNCTION TO COMPUTE PERCENTILE OF T DIST.      *
*         XBAR      -  AVERAGE OF X                                        *
*         YBAR      -  AVERAGE OF Y                                        *
*                                                                          *
*    REQUIRED EXTERNAL FUNCTIONS:  LNORM, TIN, ANORDF, ANORIN              *
*                                                                          *
****************************************************************************
*
      INTEGER REP, CASE
      REAL ALPHA, TWO
      PARAMETER(ALPHA=0.05)
      PARAMETER(TWO=2.0)
      PARAMETER(REP=1000)
      PARAMETER(CASE=18)
      INTEGER I, I1, J, U, V, XSEED, YSEED, N, M, CLOSE,
     +        A, B, A1, B1, CLOSEN
      REAL MUX, MUY, XBAR, YBAR, SIGMAX, SIGMAY, DF, R, RLHAT(REP),
     +     K, DELTA, X(120), Y(120), X1(120), Y1(120), SUMSQX,
     +     SUMSQY, SUMX, SUMY, TIN, ANORDF, RN, RM, DIFF, SIGMA, TEMP,
     +     ER, ERSUM, ERSSQ, ERBAR, ERSV,
     +     BIGU, RTILD, NU, EPS, ANORIN,ERN, ERSUMN, ERSSQN, ERBARN,
     +     ERSVN, DFF, RB(REP)
*-----------------------------------------------------------------------
      CALL EXCMS('FILEDEF 12 DISK SEQ4 DATA A1')
      CALL EXCMS('FILEDEF 18 DISK AEQ4 DATA A1')
      DO 2000 I=1, CASE
      READ (12,100) XSEED, YSEED, MUX, MUY, SIGMAX, SIGMAY, N, M, R
100   FORMAT (I5,1X,I5,1X,F5.1,1X,F7.3,1X,F4.1,1X,F4.1,1X,I2,1X,I2,1X,
     +F5.3)
      DF = REAL(N+M-2)
      RN = REAL(N)
      RM = REAL(M)
      ERSUMN = 0.0
      ERSSQN = 0.0
      ERSUM = 0.0
      ERSSQ = 0.0
      DO 1000 J=1, REP
         CALL LNORM(XSEED, X, N, 2, 0)
         CALL LNORM(YSEED, Y, M, 2, 0)
         SUMSQX = 0.0
         SUMSQY = 0.0
         SUMX = 0.0
         SUMY = 0.0
         DO 200 U= 1, N
            X1(U) = X(U) * SIGMAX + MUX
            SUMSQX= SUMSQX + X1(U) * X1(U)
            SUMX = SUMX + X1(U)
200      CONTINUE
         XBAR= SUMX / RN
         DO 300 V=1, M
            Y1(V) = Y(V) * SIGMAY + MUY
            SUMSQY = SUMSQY + Y1(V) * Y1(V)
```

```
            SUMY = SUMY + Y1(V)
300       CONTINUE
*-----------------------------------------------------------------------
* PROCEDURE FOR PARAMETRIC
          YBAR = SUMY / RM
          SIGMA = SQRT( (SUMSQX - RN*XBAR*XBAR + SUMSQY - RM*YBAR*YBAR)
     +          / DF)
          K = MAX( (XBAR - YBAR) / SIGMA, 0.0 )
          DELTA = K - SQRT( (RN+RM)/(RN*RM) + K*K / (2.0*(RN+RM-2.0)))
     +          * TIN(1.0-ALPHA,DF)
          RLHAT(J) = ANORDF(DELTA/SQRT(TWO))
          ER = R - RLHAT(J)
          ERSUM = ERSUM + ER
          ERSSQ = ERSSQ + ER * ER
*-----------------------------------------------------------------------
* PROCEDURE FOR NONPARAMETRIC
          BIGU = 0.0
          DO 500 A = 1, N
             DO 400 B = 1, M
                IF (X1(A) .GT. Y1(B)) BIGU = BIGU + 1.0
400          CONTINUE
500       CONTINUE
          RTILD = BIGU / (RN * RM)
          NU = MIN (RN, RM)
          EPS = 1.0 / SQRT(4.0 * NU) * ANORIN(1.0 - ALPHA)
          RB(J) = RTILD - EPS
          ERN = R - RB(J)
          ERSUMN = ERSUMN + ERN
          ERSSQN = ERSSQN + ERN * ERN
1000   CONTINUE
*-----------------------------------------------------------------------
          ERBARN = ERSUMN / REAL(REP)
          ERSVN = ( ERSSQN - REAL(REP) * ERBARN * ERBARN ) / REAL(REP-1)
          DFF = 2.0
          DO 1300 A1 = 1, REP
             DO 1200 B1 = A1 + 1, REP
                IF (RB(B1) .LT. RB(A1)) THEN
                   TEMP = RB(A1)
                   RB(A1) = RB(B1)
                   RB(B1) = TEMP
                ENDIF
1200         CONTINUE
          IF (((R-RB(A1)) .GE. 0.1E-6) .AND.
     +    ((R-RB(A1)) .LE. DFF)) THEN
             DFF = R -  RB(A1)
             CLOSEN = A1
          ENDIF
1300   CONTINUE
*-----------------------------------------------------------------------
* PROCEDURE OF PARAMETRIC
          ERBAR = ERSUM / REAL(REP)
          ERSV = ( ERSSQ - REAL(REP) * ERBAR * ERBAR ) / REAL(REP-1)
       DIFF = 2.0
       DO 1800 I1=1, REP
          DO 1500 J=I1+1, REP
             IF (RLHAT(J) .LT. RLHAT(I1))  THEN
```

```
                TEMP = RLHAT(I1)
                RLHAT(I1) = RLHAT(J)
                RLHAT(J) = TEMP
             ENDIF
1500      CONTINUE
          IF (((R-RLHAT(I1)) .GE. 0.1E-6) .AND.
     +    ((R-RLHAT(I1)) .LE. DIFF)) THEN
             DIFF = R -  RLHAT(I1)
             CLOSE = I1
          ENDIF
1800   CONTINUE
       WRITE (18,1900) I, N, M, R, SIGMAX
       WRITE (18,1910) RLHAT(NINT(REAL(REP)*(1.0-ALPHA))), ERBAR
       WRITE (18,1920) RB(NINT(REAL(REP)*(1.0-ALPHA))), ERBARN
1900   FORMAT('CASE: ',I2,/,'N: ', I2, T16, 'M: ', I2, /, 'R: ', F4.3,
     +        T16, 'SIGMA : ', F4.1)
1910   FORMAT('< PARAMETRIC >', /, 'RLHAT: ', F5.4, T16, 'ERROR: ', F5.4)
1920   FORMAT('< NONPARAMETRIC >', /, 'RLHAT: ', F5.4, T16, 'ERROR: ',
     +         F5.4,//)
2000   CONTINUE
       STOP
       END
```

APPENDIX D. FORTRAN CODE FOR INTERVAL ESTIMATION

PROCEDURE - NORMAL UNEQUAL VARIANCES CASE

```
      PROGRAM UNEQSM
*
************************************************************************
*     THIS PROGRAM IS TO VALIDATE THE LOWER CONFIDENCE LIMIT APPROXI-  *
*     MATION PROCEDURE FOR P( X > Y ), WHERE X, Y ARE INDEPENDENTLY    *
*     NORMALLY DISTRIBUTED WITH UNKNOWN MEANS AND UNKNOWN AND UNEQUAL  *
*     VARIANCES                                                        *
*                                                                      *
*     ALPHA    -  NOMINAL CONFIDENCE LEVEL                             *
*     ANORDF   -  IMSL FUNCTION FOR NORMAL PROBABILITY                 *
*     ASVX     -  SAMPLE VARIANCE OF X DEVIDED BY SAMPLE SIZE          *
*     ASVY     -  SAMPLE VARIANCE OF Y DEVIDED BY SAMPLE SIZE          *
*     CASE     -  NUMBER OF TESTING                                    *
*     ERROR    -  ERROR OF ESTIMATION                                  *
*     ERRBAR   -  MEAN OF ERROR                                        *
*     ERRSSQ   -  SUM OF SQUARE OF ERROR                               *
*     ERRSUM   -  SUM OF ERROR                                         *
*     ERRSV    -  SAMPLE VARIANCE OF ERROR                             *
*     CLOSE    -  INDEX OF THE CLOSEST ESTIMAT                         *
*     DELTAH   -  DELTAHAT; A ESTIMATOR OF DELTA                       *
*     DF       -  DEGREES OF FREEDOM OF DELTAHAT                       *
*     LNORM    -  RANDOM NUMBER GENERATOR FOR NORMAL VARIATES          *
*     M        -  SAMPLE SIZE OF RANDOM VARIABLE X                     *
*     MUX      -  POPULATION MEAN OF X                                 *
*     MUY      -  POPULATION MEAN OF Y                                 *
*     N        -  SAMPLE SIZE OF RANDOM VARIABLE Y                     *
*     R        -  REAL RELIABILITY                                     *
*     REP      -  REPETITION OF SIMULATIONS                            *
*     RLHAT    -  LOWER CONFIDENCE LIMIT OF RELIABILITY                *
*     SDHAT    -  SAMPLE STANDARD DEVIATION OF THE DELTAHAT            *
*     SIGMAX   -  STANDARD DEVIATION OF X                              *
*     SIGMAY   -  STANDARD DEVIATION OF Y                              *
*     SUMSQX   -  SUM OF SQUARE OF RANDOM SAMPLE OF X                  *
*     SUMSQY   -  SUM OF SQUARE OF RANDOM SAMPLE OF Y                  *
*     SUMX     -  SUM OF RANDOM SAMPLE OF X                            *
*     SUMY     -  SUM OF RANDOM SAMPLE OF Y                            *
*     SVX      -  SAMPLE VARIANCE OF X                                 *
*     SVY      -  SAMPLE VARIANCE OF Y                                 *
*     T        -  PERCENTILE OF THE T DISTRIBUTION                     *
*     TIN      -  IMSL FUNCTION TO COMPUTE PERCENTILE OF T DISTRIBUTION *
*     VARHAT   -  SAMPLE VARIANCE OF DELTAHAT                          *
*     XBAR     -  SAMPLE MEAN OF X                                     *
*     YBAR     -  SAMPLE MEAN OF Y                                     *
*                                                                      *
*     REQUIRED EXTERNAL FUNCTIONS: LNORM, TIN, ANORDF                  *
*                                                                      *
************************************************************************
*
```

```
      INTEGER REP, CASE
      REAL ALPHA
      PARAMETER(REP=1000)
      PARAMETER(CASE=18)
      PARAMETER(ALPHA=0.05)
      INTEGER I, I1, J, U, V, XSEED, YSEED, N, M, CLOSE
      REAL RM, RN, MUX, MUY, XBAR, YBAR, SIGMAX, SIGMAY, R,
     +     X(100), Y(100), X1, Y1, SUMSQX, SUMSQY, SUMX, SUMY,
     +     SVX, SVY, ASVX, ASVY, DF1, DF, T, TIN, DELTAH, VARHAT,
     +     SDHAT, RLHAT(REP), ANORDF, TEMP, DIFF, ERROR, ERRSUM, ERRSSQ,
     +     ERRBAR, ERRSV
*-----------------------------------------------------------------------
      CALL EXCMS('FILEDEF 12 DISK SETUNQ DATA A1')
      CALL EXCMS('FILEDEF 18 DISK OPT2 DATA A1')
      WRITE (18,*) 'UNEQUAL VARIANCES'
      DO 2000 I=1, CASE
      READ (12,100) XSEED, YSEED, MUX, MUY, SIGMAX, SIGMAY, N, M, R
100   FORMAT (I5,1X,I5,1X,F5.1,1X,F7.3,1X,F4.1,1X,F4.1,1X,I2,1X,I2,1X,
     +F5.3)
      RN = REAL(N)
      RM = REAL(M)
      ERRSUM = 0.0
      ERRSSQ = 0.0
*
*     < TRANSFORMATION OF X, Y TO DESIRED PROPERTIES >
*
      DO 1000 J=1, REP
         CALL LNORM(XSEED, X, N, 2, 0)
         CALL LNORM(YSEED, Y, M, 2, 0)
         SUMSQX = 0.0
         SUMSQY = 0.0
         SUMX = 0.0
         SUMY = 0.0
         DO 200 U= 1, N
            X1 = X(U) * SIGMAX + MUX
            SUMSQX= SUMSQX + X1 * X1
            SUMX = SUMX + X1
200      CONTINUE
         XBAR= SUMX / RN
         DO 300 V=1, M
            Y1 = Y(V) * SIGMAY + MUY
            SUMSQY = SUMSQY + Y1 * Y1
            SUMY = SUMY + Y1
300      CONTINUE
*
*        < COMPUTE CONFIDENCE LIMIT OF RELIABILITY >
         YBAR = SUMY / RM
         SVX = (SUMSQX - RN * XBAR * XBAR) / (RN - 1.0)
         SVY = (SUMSQY - RM * YBAR * YBAR) / (RM - 1.0)
         ASVX = SVX / RN
         ASVY = SVY / RM
         DF1 = (SVX + SVY) * (SVX + SVY) / ( SVX*SVX / (RN-1.0) +
     +         SVY*SVY / (RM-1.0) )
         DF = ANINT(DF1)
         T = TIN( 1.0-ALPHA, DF)
         DELTAH = (XBAR - YBAR) / SQRT(SVX + SVY)
```

```
      VARHAT = (ASVX + ASVY) / (SVX + SVY) +
    +          (XBAR - YBAR) * (XBAR - YBAR) /
    +          ( 2.0*(SVX + SVY)**3 ) *
    +          ( (SVX**2) / (RN-1.0) + (SVY**2) / (RM-1.0) )
      SDHAT = SQRT(VARHAT)
      RLHAT(J) = ANORDF(DELTAH - T * SDHAT)
*
*     < COMPUTE THE MEAN AND VARIANCE OF ESTIMAT. 'N ERROR >
*
      ERROR= R - RLHAT(J)
      ERRSUM = ERRSUM + ERROR
      ERRSSQ = ERRSSQ + ERROR * ERROR
      ERRBAR = ERRSUM / REAL(REP)
      ERRSV = ( ERRSSQ - REAL(REP) * ERRBAR * ERRBAR ) / REAL(REP-1)
1000 CONTINUE
*
*   < SORT CONFIDENCE LIMITS IN ASCENDING ORDER >
*
    DIFF = 2.0
    DO 1800 I1=1, REP
      DO 1500 J=I1+1, REP
        IF (RLHAT(J) .LT. RLHAT(I1))  THEN
          TEMP = RLHAT(I1)
          RLHAT(I1) = RLHAT(J)
          RLHAT(J) = TEMP
        ENDIF
1500  CONTINUE
*
*     < FIND THE CLOSEST CONFIDENCE LIMIT ESTIMATE >
*
      IF (((R-RLHAT(I1)) .GE. 0.1E-6) .AND.
    + ((R-RLHAT(I1)) .LE. DIFF)) THEN
        DIFF = R -  RLHAT(I1)
        CLOSE = I1
      ENDIF
1800 CONTINUE
    WRITE (18,1900) I, MUX, N, SIGMAX, MUY, M, SIGMAY, R,
    +                RLHAT(NINT(REAL(REP)*(1.0-ALPHA))), RLHAT(CLOSE),
    +                REAL(CLOSE)/1000.0, ERRBAR, ERRSV
1900 FORMAT('SIMULATION: ',I2,/,'MUX: ',F5.1,T16,'N: ',I2,
    +       T35,'SIGMAX: ',F4.1,/,'MUY: ',F7.3,T16,'M: ',I2,
    +       T35,'SIGMAY: ',F4.1,/,'TRUE R : ',F7.5,
    +       T35,'RLHAT: ',F7.5,/,'CLOSEST RLHAT: ',F7.5,
    +       T35,'TRUE CONFIDENCE LEVEL: ',F5.3,/,
    +       'MEAN ERROR: ',F7.5,/,
    +       'VARIANCE OF ERROR: ',F7.5,///)
2000 CONTINUE
    STOP
    END
```

APPENDIX E. SIMULATION PARAMETER SETS FOR UNEQUAL VARIANCES CASE

```
FILE:  SETUNQ DATA (FOR APPROXIMATE PROCEDURE)

16807 93943  10.0   7.133  1.0  2.0 10 20  .900   1
16807 93943  10.0   7.133  1.0  2.0 25 35  .900   2
16807 93943  10.0   7.133  1.0  2.0 75 50  .900   3
16807 93943 300.0 247.142 10.0 40.0 10 20  .900   4
16807 93943 300.0 247.142 10.0 40.0 25 35  .900   5
16807 93943 300.0 247.142 10.0 40.0 75 50  .900   6
16807 93943  10.0   6.322  1.0  2.0 10 20  .950   7
16807 93943  10.0   6.322  1.0  2.0 25 35  .950   8
16807 93943  10.0   6.322  1.0  2.0 75 50  .950   9
16807 93943 300.0 232.175 10.0 40.0 10 20  .950   10
16807 93943 300.0 232.175 10.0 40.0 25 35  .950   11
16807 93943 300.0 232.175 10.0 40.0 75 50  .950   12
16807 93943  10.0   4.799  1.0  2.0 10 20  .990   13
16807 93943  10.0   4.799  1.0  2.0 25 35  .990   14
16807 93943  10.0   4.799  1.0  2.0 75 50  .990   15
16807 93943 300.0 204.097 10.0 40.0 10 20  .990   16
16807 93943 300.0 204.097 10.0 40.0 25 35  .990   17
16807 93943 300.0 204.097 10.0 40.0 75 50  .990   18
----- ----- ----- ------- ---- ---- -- --  ----
X     Y     X     Y       X    Y
SEED  SEED  MEAN  MEAN    STDV STDV N  M    R
```

FILE: SUQ7 DATA (FOR COMPARISON SIMULATIONS).

```
16807 93943  10.0    7.133  1.0  2.0 10 15  .900   1
16807 93943  10.0    7.133  1.0  2.0 70 35  .900   2
16807 93943  10.0    7.133  1.0  2.0 90 90  .900   3
16807 93943 300.0  247.142 10.0 40.0 10 15  .900   4
16807 93943 300.0  247.142 10.0 40.0 70 35  .900   5
16807 93943 300.0  247.142 10.0 40.0 90 90  .900   6
16807 93943  10.0    6.322  1.0  2.0 10 15  .950   7
16807 93943  10.0    6.322  1.0  2.0 70 35  .950   8
16807 93943  10.0    6.322  1.0  2.0 90 90  .950   9
16807 93943 300.0  232.175 10.0 40.0 10 15  .950   10
16807 93943 300.0  232.175 10.0 40.0 70 35  .950   11
16807 93943 300.0  232.175 10.0 40.0 90 90  .950   12
16807 93943  10.0    4.799  1.0  2.0 10 15  .990   13
16807 93943  10.0    4.799  1.0  2.0 70 35  .990   14
16807 93943  10.0    4.799  1.0  2.0 90 90  .990   15
16807 93943 300.0  204.097 10.0 40.0 10 15  .990   16
16807 93943 300.0  204.097 10.0 40.0 70 35  .990   17
16807 93943 300.0  204.097 10.0 40.0 90 90  .990   18
----- ----- ----- ------- ---- ---- -- --  ----
I5    I5    F5.1  F7.3    F4.1 F4.1 I2 I2  F5.3
X     Y     X     Y       X    Y    N  M   R
SEED  SEED  MEAN  MEAN    STDV STDV
```

APPENDIX F. FORTRAN CODE FOR SIMULTANEOUS COMPARISON SIMULATION OF APPROXIMATE PROCEDURE VS. NONPARAMETRIC PROCEDURE - UNEQUAL VARIANCES

```
      PROGRAM COMUQ7
*
*************************************************************************
*    THIS PROGRAM IS TO COMPUTE THE LOWER CONFIDENCE LIMITS OF          *
*    R = P (X > Y) SIMULTANEOUSLY FOR BOTH THE APPROXIMATE PRO-         *
*    AND THE NONPARAMETRIC PROCEDURE, WHERE X, Y ARE INDEPENDENTLY      *
*    NORMALLY DISTRIBUTED WITH UNKNOWN MEANS AND UNKNOWN AND UNEQUAL    *
*    VARIANCES                                                          *
*                                                                       *
*    ALPHA   -  NOMINAL CONFIDENCE LEVEL                                *
*    ANORDF  -  IMSL FUNCTION FOR NORMAL PROBABILITY                    *
*    ANORIN  -  IMSL FUNCTION FOR INVERSE NORMAL CDF                    *
*    ASVX    -  SAMPLE VARIANCE OF X DEVIDED BY SAMPLE SIZE             *
*    ASVY    -  SAMPLE VARIANCE OF Y DEVIDED BY SAMPLE SIZE             *
*    BIGU    -  MANN-WHITNEY STATISTIC                                  *
*    CASE    -  NUMBER OF TESTING                                       *
*    EPS     -  WIDTH OF THE CONFIDENCE BOUND                           *
*    EROR    -  EROR OF ESTIMATION  (APPROXI PROCEDURE)                 *
*    ERORN   -  EROR OF ESTIMATION  (NONPARA PROCEDURE)                 *
*    ERBAR   -  MEAN OF EROR (APPROXI PROCEDURE)                        *
*    ERBARN  -  MEAN OF EROR (NONPARA PROCEDURE)                        *
*    ERSSQ   -  SUM OF SQUARE OF EROR (APPROXI PROCEDURE)               *
*    ERSSQN  -  SUM OF SQUARE OF EROR (NONPARA PROCEDURE)               *
*    ERSUM   -  SUM OF EROR (APPROXI PROCEDURE)                         *
*    ERSUMN  -  SUM OF EROR (NONPARA PROCEDURE)                         *
*    ERSV    -  SAMPLE VARIANCE OF EROR (APPROXI PROCEDURE)             *
*    ERSVN   -  SAMPLE VARIANCE OF EROR (NONPARA PROCEDURE)             *
*    CLOSE   -  INDEX OF THE CLOSEST ESTIMAT (APPROXI PROCEDURE)        *
*    CLOSEN  -  INDEX OF THE CLOSEST ESTIMAT (NONPARA PROCEDURE)        *
*    DELTAH  -  DELTAHAT; A ESTIMATOR OF DELTA                          *
*    DF      -  DEGREES OF FREEDOM OF DELTAHAT                          *
*    LNORM   -  RANDOM NUMBER GENERATOR FOR NORMAL VARIATES             *
*    M       -  SAMPLE SIZE OF RANDOM VARIABLE X                        *
*    MUX     -  POPULATION MEAN OF X                                    *
*    MUY     -  POPULATION MEAN OF Y                                    *
*    N       -  SAMPLE SIZE OF RANDOM VARIABLE Y                        *
*    NU      -  THE SMALLER OF SAMPLE SIZES                             *
*    R       -  REAL RELIABILITY                                        *
*    RB      -  CONFIDENCE BOUND OF THE RELIABILITY                     *
*    REP     -  REPETITION OF SIMULATIONS                               *
*    RLHAT   -  LOWER CINFIDENCE LIMIT OF RELIABILITY                   *
*    RTILD   -  POINT ESTIMATOR OF THE RELIABILITY                      *
*    SDHAT   -  SAMPLE STANDARD DEVIATION OF THE DELTAHAT               *
*    SIGMAX  -  STANDARD DEVIATION OF X                                 *
*    SIGMAY  -  STANDARD DEVIATION OF Y                                 *
*    SUMSQX  -  SUM OF SQUARE OF RANDOM SAMPLE OF X                     *
```

```
*     SUMSQY    -  SUM OF SQUARE OF RANDOM SAMPLE OF Y                           *
*     SUMX      -  SUM OF RANDOM SAMPLE OF X                                     *
*     SUMY      -  SUM OF RANDOM SAMPLE OF Y                                     *
*     SVX       -  SAMPLE VARIANCE OF X                                          *
*     SVY       -  SAMPLE VARIANCE OF Y                                          *
*     T         -  PERCENTILE OF THE T DISTRIBUTION                              *
*     TIN       -  IMSL FUNCTION TO COMPUTE PERCENTILE OF T DISTRIBUTION         *
*     VARHAT    -  SAMPLE VARIANCE OF DELTAHAT                                   *
*     XBAR      -  SAMPLE MEAN OF X                                              *
*     YBAR      -  SAMPLE MEAN OF Y                                              *
*                                                                                *
*     REQUIRED EXTERNAL FUNCTIONS: LNORM, TIN, ANORDF, ANORIN                    *
*                                                                                *
**********************************************************************************
*
      INTEGER REP, CASE
      REAL ALPHA
      PARAMETER(REP=1000)
      PARAMETER(CASE=18)
      PARAMETER(ALPHA=0.05)
      INTEGER I, I1, J, U, V, XSEED, YSEED, N, M, CLOSE,
     +        A, B, A1, B1, CLOSEN
      REAL RM, RN, MUX, MUY, XBAR, YBAR, SIGMAX, SIGMAY, R,
     +     X(100), Y(100), X1(100), Y1(100), SUMSQX, SUMSQY, SUMX, SUMY,
     +     SVX, SVY, ASVX, ASVY, DF1, DF, T, TIN, DELTAH, VARHAT,
     +     SDHAT, RLHAT(REP), ANORDF, TEMP, DIFF, EROR, ERSUM, ERSSQ,
     +     ERBAR, ERSV,
     +     BIGU, RTILD, NU, EPS, ANORIN, ERN, ERSUMN, ERSSQN, ERBARN,
     +     ERSVN, DFF, RB(REP)
*-------------------------------------------------------------------------------
      CALL EXCMS('FILEDEF 12 DISK SUQ7 DATA A1')
      CALL EXCMS('FILEDEF 18 DISK AUQ7 DATA A1')
      WRITE (18,*) 'UNEQUAL VARIANCES'
      DO 2000 I=1, CASE
      READ (12,100) XSEED, YSEED, MUX, MUY, SIGMAX, SIGMAY, N, M, R
100   FORMAT (I5,1X,I5,1X,F5.1,1X,F7.3,1X,F4.1,1X,F4.1,1X,I2,1X,I2,1X,
     +F5.3)
      RN = REAL(N)
      RM = REAL(M)
      ERSUM = 0.0
      ERSSQ = 0.0
      ERSUMN = 0.0
      ERSSQN = 0.0
      DO 1000 J=1, REP
         CALL LNORM(XSEED, X, N, 2, 0)
         CALL LNORM(YSEED, Y, M, 2, 0)
         SUMSQX = 0.0
         SUMSQY = 0.0
         SUMX = 0.0
         SUMY = 0.0
         DO 200 U= 1, N
            X1(U) = X(U) * SIGMAX + MUX
            SUMSQX= SUMSQX + X1(U) * X1(U)
            SUMX = SUMX + X1(U)
200      CONTINUE
         XBAR= SUMX / RN
```

```
      DO 300 V=1, M
         Y1(V) = Y(V) * SIGMAY + MUY
         SUMSQY = SUMSQY + Y1(V) * Y1(V)
         SUMY = SUMY + Y1(V)
300   CONTINUE
      YBAR = SUMY / RM
      SVX = (SUMSQX - RN * XBAR * XBAR) / (RN - 1.0)
      SVY = (SUMSQY - RM * YBAR * YBAR) / (RM - 1.0)
      ASVX = SVX / RN
      ASVY = SVY / RM
      DF1 = (SVX + SVY) * (SVX + SVY) / ( SVX*SVX / (RN-1.0) +
     +        SVY*SVY / (RM-1.0) )
      DF = ANINT(DF1)
      T = TIN( 1.0-ALPHA, DF)
      DELTAH = (XBAR - YBAR) / SQRT(SVX + SVY)
      VARHAT = (ASVX + ASVY) / (SVX + SVY) +
     +         (XBAR - YBAR) * (XBAR - YBAR) /
     +         ( 2.0*(SVX + SVY)**3 ) *
     +         ( (SVX**2) / (RN-1.0) + (SVY**2) / (RM-1.0) )
      SDHAT = SQRT(VARHAT)
      RLHAT(J) = ANORDF(DELTAH - T * SDHAT)
      EROR = R - RLHAT(J)
      ERSUM = ERSUM + EROR
      ERSSQ = ERSSQ + EROR * EROR
*-----------------------------------------------------------------------
* PROCEDURE FOR NONPARAMETRIC
      BIGU = 0.0
      DO 500 A = 1, N
         DO 400 B = 1, M
            IF (X1(A) .GT. Y1(B)) BIGU = BIGU + 1.0
400      CONTINUE
500   CONTINUE
      RTILD = BIGU / (RN * RM)
      NU = MIN (RN, RM)
      EPS = 1.0 / SQRT(4.0 * NU) * ANORIN(1.0 - ALPHA)
      RB(J) = RTILD - EPS
      ERN = R - RB(J)
      ERSUMN = ERSUMN + ERN
      ERSSQN = ERSSQN + ERN * ERN
1000  CONTINUE
*-----------------------------------------------------------------------
      ERBARN = ERSUMN / REAL(REP)
      ERSVN = ( ERSSQN - REAL(REP) * ERBARN * ERBARN ) / REAL(REP-1)
      DFF = 2.0
      DO 1300 A1 = 1, REP
         DO 1200 B1 = A1 + 1, REP
            IF (RB(B1) .LT. RB(A1)) THEN
               TEMP = RB(A1)
               RB(A1) = RB(B1)
               RB(B1) = TEMP
            ENDIF
1200     CONTINUE
      IF (((R-RB(A1)) .GE. 0.1E-6) .AND.
     +    ((R-RB(A1)) .LE. DFF)) THEN
         DFF = R -  RB(A1)
         CLOSEN = A1
```

```
         ENDIF
1300  CONTINUE
*-----------------------------------------------------------------------
* PROCEDURE OF PARAMETRIC
      ERBAR = ERSUM / REAL(REP)
      ERSV = ( ERSSQ - REAL(REP) * ERBAR * ERBAR ) / REAL(REP-1)
      DIFF = 2.0
      DO 1800 I1=1, REP
         DO 1500 J=I1+1, REP
            IF (RLHAT(J) .LT. RLHAT(I1))  THEN
               TEMP = RLHAT(I1)
               RLHAT(I1) = RLHAT(J)
               RLHAT(J) = TEMP
            ENDIF
1500     CONTINUE
         IF (((R-RLHAT(I1)) .GE.  0.1E-6) .AND.
     +   ((R-RLHAT(I1)) .LE.  DIFF)) THEN
            DIFF = R -  RLHAT(I1)
            CLOSE = I1
         ENDIF
1800  CONTINUE
      WRITE (18,1900) I, N, M, R, SIGMAX, SIGMAY
      WRITE (18,1910) RLHAT(NINT(REAL(REP)*(1.0-ALPHA))), ERBAR
      WRITE (18,1920) RB(NINT(REAL(REP)*(1.0-ALPHA))), ERBARN
1900  FORMAT('CASE: ',I2,/,'N: ', I2, T16, 'M: ', I2, /, 'R: ', F4.3,
     +       T16, 'SIGMA X: ', F4.1, T35, 'SIGMA Y: ',F4.1)
1910  FORMAT('< PARAMETRIC >', /, 'RLHAT: ', F5.4, T16, 'ERROR: ', F5.4)
1920  FORMAT('< NONPARAMETRIC >', /, 'RLHAT: ', F5.4, T16, 'ERROR: ',
     +        F5.4,//)
2000  CONTINUE
      STOP
      END
```

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